Solutions Exam Signals and Systems

21 january 2016

Problem 1: signals and spectra

(a) The first signal has amplitude 2 and makes three oscillations in two 2 seconds, so its frequency is 1.5Hz. It is clearly a cosine which is reflected in the time-axis, so the phase is π. We conclude $x(t) = 2\cos(3\pi t + \pi)$.

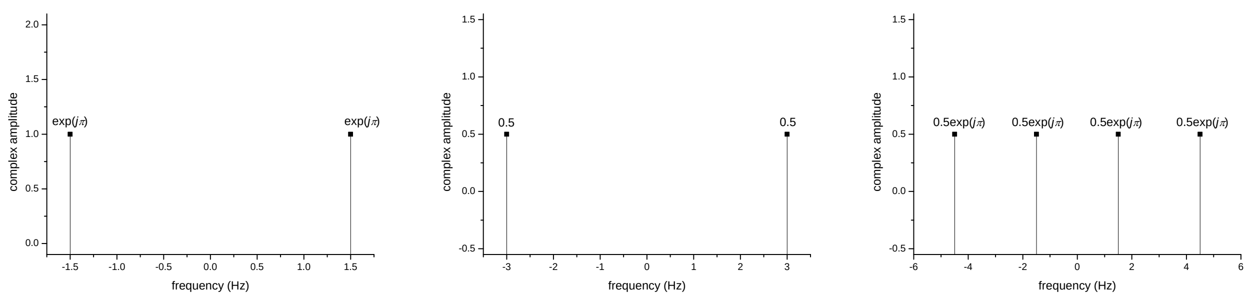
The second signal is a cosine with amplitude 1, and frequency 6Hz, so $y(t) = \cos(12\pi t)$.

Careful inspection of the third plot shows that it is an AM signal that is constructed from the other two plots: $z(t) = x(t)y(t) = 2\cos(3\pi t + \pi)\cos(12\pi t)$. This can be rewritten (formula 9 of the formula sheet) as $z(t) = \cos(9\pi t - \pi) + \cos(15\pi t + \pi) = \cos(9\pi t + \pi) + \cos(15\pi t + \pi)$.

(b) We use the inverse Euler formula $cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$.

$$\begin{aligned}
x(t) &= -2\cos(3\pi t) = 2\cos(3\pi t + \pi) = e^{j\pi}e^{j\pi 3t} + e^{j\pi}e^{-j\pi 3t} \\
y(t) &= \sin(6\pi t + \pi/2) = \cos(6\pi t) = \frac{1}{2}e^{j\pi 6t} + \frac{1}{2}e^{-j\pi 6t} \\
z(t) &= x(t)y(t) = \frac{e^{j\pi}}{2}e^{-j\pi 9t} + \frac{e^{j\pi}}{2}e^{j\pi 9t} + \frac{e^{j\pi}}{2}e^{-j\pi 3t} + \frac{e^{j\pi}}{2}e^{j\pi 3t}
\end{aligned}$$

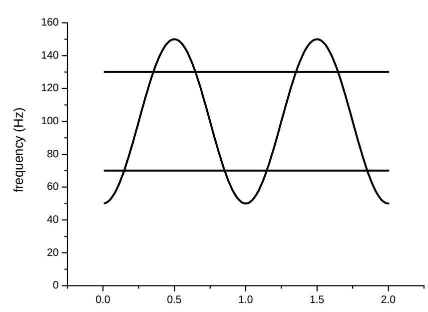
(c) The spectra of the signals are shown in the following three plots:



(d) First, we need to rewrite $x(t)$ as a sum of terms:

$$x(t) = \frac{1}{2}\cos(2\pi 70t) + \frac{1}{2}\cos(2\pi 130t) + \cos(2\pi 100t - 50\sin(2\pi t))$$

The first two terms are simply cosines with the frequencies 70 and 130Hz. The last term is a Frequency Modulated (FM) signal. Its instantaneous frequency (in Hz) is the derivative of the angle function divided by 2π i.e. $f_i = 100 - 50\cos(2\pi t)$. Now, we can plot the spectrogram:



Problem 2: Fourier analysis

(a) According to the Fourier synthesis formula (using $T_0 = 2$):

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j\pi kt} = 3e^{-j3\pi t} + 2e^{-j\pi/2}e^{-j\pi t} + 2e^{j\pi/2}e^{j\pi t} + 3e^{j3\pi t} \\ &= 4\cos(2\pi 0.5t + \pi/2) + 6\cos(2\pi 1.5t) \end{aligned}$$

So, $DC = 0$, $A = 4$, $f_0 = 0.5$, $\phi_0 = \pi/2$, $B = 6$, $f_1 = 1.5$, and $\phi_1 = 0$.

(b) Again, we use the Fourier synthesis formula (for all t):

$$g(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\pi kt/T_0} = f(t-d) = \sum_{k=-\infty}^{\infty} a_k e^{j\pi k(t-d)/T_0} = \sum_{k=-\infty}^{\infty} (a_k \cdot e^{-j\pi kd/T_0}) e^{j\pi kt/T_0}$$

Hence, we find $b_k = a_k \cdot e^{-j\pi kd/T_0}$.

(c) According to the Fourier analysis formula (with $T_0 = 2$) we find for the DC-term (i.e. $k = 0$):

$$2a_0 = \int_0^1 t\ dt + \int_1^2 1\ dt = \left[t^2/2\right]_0^1 + [t]_1^2 = \left(\frac{1}{2} - 0 + 2 - 1\right) = 1\frac{1}{2}$$

So, $a_0 = 3/4$. For other k we find (where $\alpha = -j\pi k$):

$$2a_k = \int_0^1 t \cdot e^{-j(2\pi/2)kt}\ dt + \int_1^2 1 \cdot e^{-j(2\pi/2)kt}\ dt = \int_0^1 t \cdot e^{\alpha t}\ dt + \int_1^2 e^{\alpha t}\ dt$$

Using the standard integrals from the formula sheet this reduces to:

$$2a_k = \left[\frac{\alpha t - 1}{\alpha^2} e^{\alpha t}\right]_0^1 + \left[\frac{e^{\alpha t}}{\alpha}\right]_1^2 = \frac{(\alpha - 1)e^{\alpha} + 1}{\alpha^2} + \frac{e^{2\alpha} - e^{\alpha}}{\alpha} = \frac{\alpha e^{\alpha} - e^{\alpha} + 1 + \alpha e^{2\alpha} - \alpha e^{\alpha}}{\alpha^2}$$

Now, we use $e^{2\alpha} = 1$, and $e^{\alpha} = (-1)^k$:

$$2a_k = \frac{1 - e^{\alpha} + \alpha e^{2\alpha}}{\alpha^2} = \frac{1 - (-1)^k + \alpha}{\alpha^2}$$

For even k this yields $2a_k = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha} = \frac{1}{-j\pi k}$, so $a_k = \frac{1}{-j2\pi k} = \frac{j}{2\pi k}$.

For odd k this yields $2a_k = \frac{2+\alpha}{\alpha^2} = \frac{2}{\alpha^2} + \frac{1}{\alpha} = \frac{-2}{\pi^2 k^2} + \frac{j}{\pi k}$, so $a_k = \frac{j}{2\pi k} - \frac{1}{\pi^2 k^2}$.

(d) The key insight is that $z(t)$ is the same as $x(t) - y(t)$ shifted by half a period (i.e. 1 second). So, we can use the linearity of the Fourier integral and the theorem that was proved in part (b). So, for the Fourier coefficients c_k of $z(t)$ we find: $c_k = (a_k - b_k) \cdot e^{-j\pi k}$. This yields

$$c_k = \begin{cases} \frac{3}{4} - \frac{1}{4} = \frac{1}{2} & \text{for } k = 0 \\ 0 & \text{for even } k \neq 0 \\ (\frac{j}{2\pi k} - \frac{1}{2j\pi k}) \cdot (-1)^k = \frac{1}{j\pi k} & \text{for odd } k \neq 0 \end{cases}$$

(e) First we rewrite the signal as $z(t) = 1 + \cos(2\pi 75t) + \cos(2\pi 125t)$. Now we can find the fundamental frequency $f_0 = \gcd(75, 125) = 25$Hz. So, the cases are $k = 0$, $k = \pm 3$ and $k = \pm 5$. Both components have phase angle 0, so we find:

$$a_k = \begin{cases} 1 & \text{for } k = 0, k = \pm 3, k \pm 5 \\ 0 & \text{for all other } k \end{cases}$$

Problem 3: LTI-systems

(a) First we consider the system $y_0[n] = x[n-2] + x[2-n]$. Since $y_0[0] = x[-2] + x[2]$, it is clearly not causal. It is also not time invariant, since $y_1[n-d] = x[n-d-2] + x[2-n+d] \neq x[n-d-2] + x[2-n-d]$. The system is linear, which is easy to prove:

$$\begin{aligned} & (a \cdot x + b \cdot y)[n-2] + (a \cdot x + b \cdot y)[2-n] \\ = \; & a \cdot x[n-2] + b \cdot y[n-2] + a \cdot x[2-n] + b \cdot y[2-n] \\ = \; & a(x[n-2] + x[2-n]) + b(y[n-2] + y[2-n]) \end{aligned}$$

The system $y_1[n] = x[n-2] - 8x[n-2]$ is a standard FIR filter. Hence, it is causal, linear, and time invariant.

(b) Clearly, $x[n] = u[n] - u[n-4] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-4] = [1, 1, 1, 1]$. The output is obtained by convolving this with $h = [1, 2, 3, 1]$, so we find

$$\begin{aligned} y[n] &= [1, 1, 1, 1] * [1, 2, 3, 1] = [1, 3, 6, 7, 6, 4, 1] \\ &= \delta[n] + 3\delta[n-1] + 6\delta[n-2] + 7\delta[n-3] + 6\delta[n-4] + 4\delta[n-5] + \delta[n-6] \end{aligned}$$

(c) Let us assume that the system is a FIR. This means that there is a kernel h for which $|h| = |y| - |x| + 1 = 7 - 5 + 1 = 3$. So, let $h = [a, b, c]$ and compute the convolution $h * x$:

$$\begin{array}{ccccccc} a & b & c & & & & \\ 3 & 1 & 4 & 1 & 5 & & \\ \hline 3a & 3b & 3c & & & & \\ & a & b & c & & & \\ & & 4a & 4b & 4c & & \\ & & & a & b & c & \\ & & & & 5a & 5b & 5c \\ \hline 6 & 2 & 5 & 1 & 6 & -1 & -5 \end{array}$$

From the first column, we conclude $a = 2$. From the last column, we conclude $c = -1$. Next, we use $3b + a = 2$, to deduce that $b = 0$. Checking yields $3c + b + 4a = -3 + 0 + 8 = 5$, $c + 4b + a = - - 1 + 0 + 2 = 1$, $4c + b + 5a = -4 + 0 + 10 = 6$, and $c + 5b = -1 + 0 = -1$. So, we found $h = [2, 0, -1]$ and the filter is indeed a FIR filter.

(d) Just like in part (c), assume that $h = [a, b, c]$ and compute the convolution:

$$\begin{array}{ccccccc} a & b & c & & & & \\ 1 & 2 & 3 & 2 & 1 & & \\ \hline a & b & c & & & & \\ & 2a & 2b & 2c & & & \\ & & 3a & 3b & 3c & & \\ & & & 2a & 2b & 2c & \\ & & & & a & b & c \\ \hline -1 & 0 & 2 & 2 & 0 & 0 & -1 \end{array}$$

Clearly, from the first and the last column, we conclude $a = c = -1$. From $b + 2a = 0$, we conclude $b = 2$. However, this contradicts $c + 2b + 3a = -1 + 4 - 3 = 0 \neq 2$. Hence, the filter is not a FIR filter.

(e) We can obtain the unit pulse $\delta[n]$ by adding the first two given inputs, and subtract 3 times the 3rd input shifted by 2 samples. So, we can get the output of the unit pulse as follows: $y[n] = [12, 10, 10, 24, 10] + [-8, -8, 2, 6, 2] - 3[0, 0, 4, 10, 4] = [4, 2, 0, 0, 0]$
We conclude $h = [4, 2] = 4\delta[n] + 2\delta[n-1]$.

Problem 4: frequency responses and z-transforms

(a) For the first order difference, we have $h = [1, -1]$. Hence, $H(z) = 1 - z^{-1}$ and $H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$.

(b) We rewrite the frequency response in normal-form using Euler's formula:

$$H_0(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(3 - 2\cos\hat{\omega}) = e^{-j\hat{\omega}}(3 - e^{j\hat{\omega}} - e^{-j\hat{\omega}}) = 3e^{-j\hat{\omega}} - 1 - e^{-j2\hat{\omega}} = -1 + 3e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$$

So, we find $h = [-1, 3, -1]$ and $y[n] = -x[n] + 3x[n-1] - x[n-2]$. Since $3 - 2\cos\hat{\omega} > 0$, no frequencies are completely nulled by this system.

(c) For the DC-component, we find the gain $3 - 2\cos 0 = 1$, so the DC-component is not changed. For the frequency $\hat{\omega} = \pi/3$ we find the gain $3 - 2\cos(\pi/3) = 2$ and the phase change $-\pi/3$. For the frequency $\hat{\omega} = \pi/4$ we find the gain $3 - 2\cos(\pi/4) = 3 - \sqrt{2}$ and the phase change $-\pi/4$. So, we find the output $y[n] = 5 + 6\cos\left(\frac{(n-1)\pi}{3}\right) + (6 - 2\sqrt{2})\sin\left(\frac{(n-1)\pi}{4}\right) = 5 + 6\cos\left(\frac{(n-1)\pi}{3}\right) + (6 - 2\sqrt{2})\cos\left(\frac{(n-3)\pi}{4}\right)$.

(d) A DC-component is removed by the first order difference. We can remove a frequency $\hat{\omega}$ by a system with system function $(1 - e^{j\hat{\omega}}z^{-1})(1 - e^{-j\hat{\omega}}z^{-1}) = 1 - 2\cos(\hat{\omega})z^{-1} + z^{-2}$. A first order difference will remove the DC-term. So, the filter asked for has the impulse response $[1, -1] * [1, -2\cos(5\pi/6), 1] = [1, -1] * [1, \sqrt{3}, 1] = [1, \sqrt{3} - 1, 1 - \sqrt{3}, -1]$. Its difference equation and system function are:

$$\begin{aligned} y_1[n] &= x[n] + (\sqrt{3} - 1)x[n-1] + (1 - \sqrt{3})x[n-2] - x[n-3] \\ H_1(z) &= 1 + (\sqrt{3} - 1)z^{-1} + (1 - \sqrt{3})z^{-2} - z^{-3} \end{aligned}$$

(e) Clearly, F_2 is a 12 points-averager. Since 12 is 4×3 and 2×6, this means that the cosine terms are completely nulled. The DC-component remains, so the output is simply $y[n] = 5$.